

Nowhere-zero flows in signed series-parallel graphs

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Abstract

Bouchet conjectured in 1983 that each signed graph that admits a nowhere-zero flow has a nowhere-zero 6-flow. We prove that the conjecture is true for all signed series-parallel graphs. Unlike the unsigned case, the restriction to series-parallel graphs is nontrivial; in fact, the result is tight for infinitely many graphs.

1 Introduction

A *signed graph* (G, σ) is a graph G together with a *signature*, a mapping $\sigma : E(G) \rightarrow \{+1, -1\}$, that assigns each edge with a sign. The graph G is called the *underlying graph* of (G, σ) . In this work we focus on signed graphs whose underlying graph is a *series-parallel graph*, that is, a graph that can be obtained from copies of K_2 by iterated series and parallel connections.

A signed graph can be given an *orientation* as follows. Viewing each edge as composed of two half-edges, we orient each half-edge independently; it is required that of the two half-edges of an edge e , exactly one points to its endvertex if e is positive, while none or both of them point to their endvertices if e is negative. A *nowhere-zero k -flow* (D, ϕ) on a signed graph (G, σ) is an orientation D of edges of (G, σ) and a valuation ϕ of its arcs by non-zero integers whose absolute value is smaller than k , such that for every vertex the sum of the incoming values (the *inflow*) is equal to the sum of the outgoing ones (the *outflow*). Graphs (signed or unsigned) admitting at least one nowhere-zero k -flow (for some k) are called *flow-admissible*.

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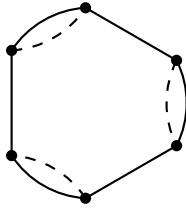


Figure 1: A signed series-parallel graph with flow number 6.

With respect to nowhere-zero flows, all-positive signed graphs (i.e., those with all edge signs positive) behave like ordinary unsigned graphs. Thus, problems about nowhere-zero flows in signed graphs include the celebrated 5-flow conjecture of Tutte [5]:

Conjecture 1 (Tutte). *Every flow-admissible graph has a nowhere-zero 5-flow.*

While there are examples showing that the analogue of Conjecture 1 is false for general signed graphs (as discussed below), Bouchet [1] conjectured that things do not get much worse:

Conjecture 2 (Bouchet). *Every flow-admissible signed graph has a nowhere-zero 6-flow.*

The best published partial result in the direction of Bouchet’s conjecture is a 30-flow theorem by Zýka [7] from 1987. Recently, a 12-flow theorem was announced by DeVos [2].

An infinite family of signed graphs reaching the bound stated in Conjecture 2 was found by Schubert and Steffen [4]. The smallest member of the family is shown in Figure 1. Interestingly, the underlying graphs of the members of this family are series-parallel. This is in sharp contrast with the situation in the unsigned case, where each flow-admissible series-parallel graph trivially admits a nowhere-zero 3-flow. In this paper, we concentrate on the family of signed series-parallel graphs and prove the corresponding restriction of Bouchet’s conjecture:

Theorem 3. *Every flow-admissible signed series-parallel graph has a nowhere-zero 6-flow.*

The proof is given in Section 5.

2 Preliminaries

In this section, we review the necessary terminology. Our graphs may contain parallel edges. The *switching at a vertex v* of a signed graph is the operation of inverting the signs on all the edges incident with v . Two signed graphs are *switching equivalent* if one can be obtained from the other by a finite sequence of

switchings. Since switching does not affect the existence of a nowhere-zero k -flow (for any k), we may treat switching equivalent graphs as identical.

It is well known that an unsigned graph is flow-admissible if and only if it is bridgeless. Before we state a corresponding characterisation for signed graphs due to Bouchet [1], we recall several basic notions. A *balanced cycle* is a cycle with an even number of negative edges. An *unbalanced cycle* is a cycle with an odd number of negative edges. A signed graph is called *unbalanced* if it contains an unbalanced cycle. Otherwise it is *balanced*. A *barbell* in a signed graph G is the union of two edge-disjoint unbalanced cycles C_1, C_2 and a path P satisfying one of the following properties:

- C_1 and C_2 are vertex-disjoint, P is internally vertex-disjoint from $C_1 \cup C_2$ and shares an endvertex with each C_i , or
- $V(C_1) \cap V(C_2)$ consists of a single vertex w , and P is the trivial path consisting of w .

A *signed circuit* in G is either a balanced cycle or a barbell in G . With respect to flows, signed circuits are analogous to cycles in unsigned graphs. The following characterisation theorem is due to Bouchet [1, Proposition 3.1]:

Theorem 4. *A signed graph G is flow-admissible if and only if each of its edges is contained in a signed circuit.*

A *two-terminal graph* (G, s, t) consists of a graph G together with two distinguished vertices, the *source terminal* s and the *target terminal* t , where $s \neq t$. We abbreviate (G, s, t) to G .

Let G_1, \dots, G_n be two-terminal graphs and H their disjoint union. The *series connection* $\mathcal{S}(G_1, \dots, G_n)$ of G_1, \dots, G_n is obtained from H by identifying, for $i = 1, \dots, n-1$, the target terminal of G_i with the source terminal of G_{i+1} . By definition, the source and target terminal of $\mathcal{S}(G_1, \dots, G_n)$ is the source terminal of G_1 and the target terminal of G_n , respectively.

The *parallel connection* $\mathcal{P}(G_1, \dots, G_n)$ of G_1, \dots, G_n is obtained from H by identifying their source terminals and identifying their target terminals. The source terminal of the resulting graph is the vertex obtained by the identification of the source terminals of the graphs G_i , and similarly for the target terminal.

A *series-parallel graph* is a two-terminal graph obtained by a sequence of series and parallel connections, starting with copies of K_2 (with some choice of the terminals).

If G_1, \dots, G_n ($n \geq 2$) are series-parallel graphs such that G is a series or a parallel connection of G_1, \dots, G_n and n is maximum with this property, then we refer to the G_i as *parts* of G . In addition, in the case of a series connection, G_1 and G_n are the *endparts* of G . We say that G' is a *piece* of G if there is a sequence $G' = H_0, H_1, \dots, H_m = G$ such that for each $j = 0, \dots, m-1$, H_j is a part of H_{j+1} . In particular, G itself is a piece of G .

The *replacement* of G' by a series-parallel graph H' in G consists in removing all the edges and non-terminal vertices of G' in G , adding H' and identifying each of its terminals with the corresponding terminal of G' in G .

We introduce the following notation for small signed series-parallel graphs: K_2^+ denotes the positive K_2 , K_2^- stands for the negative K_2 , and D is the unbalanced 2-cycle.

We define the *depth* $\text{dp}(G)$ of a signed series-parallel graph G by letting $\text{dp}(K_2^+) = \text{dp}(K_2^-) = 0$, and

$$\text{dp}(G) = 1 + \max_H \text{dp}(H),$$

where H ranges over all parts of G .

The following observation is immediate from the definition:

Observation 5. *If $0 \leq k \leq \text{dp}(G)$, then G contains a piece of depth k .*

Another useful observation is the following one; it can be proved by straightforward induction:

Observation 6. *Each non-terminal vertex of a series-parallel graph has at least two distinct neighbours.*

We will need the following lemma, proved in [3] by induction:

Lemma 7. *If $e = xy$ is an edge in a 2-connected series-parallel graph G , then (G, x, y) is a series-parallel graph.*

3 Reduced graphs

We begin by using easy reductions to prove the following observation:

Lemma 8. *Let G be a counterexample to Theorem 3 of minimum size. Then G has the following properties:*

- (i) *if G is of series type, then its endpoints are unbalanced,*
- (ii) *the degree of each non-terminal vertex is at least three,*
- (iii) *if a terminal vertex has degree two, then it is contained in a 2-cycle.*

Proof. We prove (i). Suppose that G has a balanced endpoint H ; let u denote the terminal u of H that is a cutvertex of G , and let H' be obtained from G by removing $V(H) - \{u\}$. We show using Theorem 4 that H' is flow-admissible. Suppose the contrary; then an edge e of H' is not contained in a signed circuit of H' . However, G is flow-admissible, and it is not hard to see that any signed circuit of G missing in H' is a balanced circuit contained in H (as H is balanced

and u is a cut-vertex). This is a contradiction, so H' is indeed flow-admissible. By the minimality of G , H' admits a nowhere-zero 6-flow. Since any unsigned series-parallel graph admits a nowhere-zero 3-flow, so does the balanced signed graph H . Combining these two flows, we obtain a nowhere-zero 6-flow on G , a contradiction.

Let us prove (ii). Let u be a non-terminal vertex of degree 2, say incident with edges e_1, e_2 . Switching at u if necessary, we may assume that e_1 is positive. By Observation 6, the endvertices of e_1, e_2 different from u are distinct. Contracting e_1 , we therefore obtain a (loopless) series-parallel graph G' . Since the contraction of a positive edge preserves the existence of a nowhere-zero flow, G' is flow-admissible and hence it has a nowhere-zero 6-flow by the minimality of G . This corresponds to a 6-flow ψ on G , possibly with $\psi(e_1) = 0$. However, since u has degree 2, we have $|\psi(e_1)| = |\psi(e_2)|$, so ψ is nowhere-zero.

A similar argument works for terminal vertices of degree 2 with two distinct neighbours. This proves (iii). \square

The following result will be useful in the proof of Lemmas 10 and 17 below.

Lemma 9. *If a signed series-parallel graph G' is of series type and has unbalanced endparts, then every edge of G' is contained in a barbell.*

Proof. Let the endparts of G' be denoted by R_1, R_2 . For $i = 1, 2$, let r_i be the terminal of R_i that is not a terminal of G . Let C_i ($i = 1, 2$) be an unbalanced cycle in R_i .

Let e be an edge of G' ; we need to show that e is contained in a barbell. If e is not contained in an endpart, then G' contains a path P from r_1 to r_2 containing e . Extending P to a path joining C_1 to C_2 , we obtain a barbell in G' containing e .

We may therefore assume that e is contained in R_1 . Since R_1 is unbalanced, R_1 is different from a signed K_2 and therefore 2-connected. Choose two vertex-disjoint paths in R_1 connecting the endvertices of e to C_1 . Taking the union of these paths with e and a suitable subpath of C_1 , we obtain an unbalanced cycle C'_1 in R_1 containing e . Then the union of C'_1 , C_2 and an appropriate extension of P is a barbell in G' containing e . \square

Lemma 10. *If G is a counterexample to Theorem 3 of minimum size, then G contains no pair of parallel edges of the same sign.*

Proof. Suppose that G contains parallel edges e, f of the same sign, say both positive. By Theorem 4, each edge of G is contained in a signed circuit. Thus, each edge e' of $G - e$ is clearly also contained in a signed circuit of $G - e$ unless $e' = f$.

Assume first that f is also contained in a signed circuit of $G - e$. Then $G - e$ is flow-admissible. By the minimality of G , $G - e$ admits a nowhere-zero 6-flow

φ . Adding to φ a suitable nowhere-zero 6-flow on the 2-cycle C comprised of e and f , we obtain a nowhere-zero 6-flow in G .

It follows that f is not contained in a signed circuit. Note that in this case, $G - e - f$ is flow-admissible. If we can show that each component of $G - e - f$ is series-parallel, then by the minimality of G , $G - e - f$ admits a nowhere-zero 6-flow, which is easily extended to G using a suitable flow on the above 2-cycle C .

It remains to prove that $G - e - f$ is comprised of series-parallel components. Let the terminals of G be u and v . Suppose first that $G - e$ is 2-connected. Then $G - e - f$ is connected. By Lemma 7, $(G - e - f, x, y)$ is a series-parallel graph, where x and y are the endvertices of f .

We can therefore assume that $G - e$ is not 2-connected. We claim that G is not 2-connected; assume the contrary. Since, clearly, $G - e$ is different from K_2 , there is a cutvertex of $G - e$ separating the endvertices of e . These are, however, connected by the edge f , a contradiction.

Since G is not 2-connected, it has unbalanced endpoints by Lemma 8(i), and the same holds for $G - e$. By Lemma 9, f is contained in a barbell in $G - e$, a contradiction with the hypothesis that f is not contained in a signed circuit of $G - e$. This concludes the proof. \square

Let us call a graph *reduced* if the degree of each of its non-terminal vertices is at least 3, and there is no pair of parallel edges of the same sign. Observe that each part (and hence, each piece) of a reduced graph is reduced. It is easy to see that the only reduced graph of depth 1 is the unbalanced 2-cycle D .

A *string* is a series connection of copies of K_2^+ and D where each non-terminal vertex is contained in a 2-cycle (see Figure 2). Thus, any string is a reduced graph. A string is *nontrivial* if it contains more than two vertices.

Lemma 11. *Each reduced signed series-parallel graph of depth at most 2 is switching equivalent to a string.*

Proof. Let G be a graph satisfying the assumption. If the depth of G is 0 or 1, then G is K_2^+ , K_2^- or D and the assertion holds. Assume that the depth of G is 2. Since the only reduced graph of depth 1 is D , G is necessarily of series type; since each of its parts is reduced, G is a series connection of copies of K_2^+ , K_2^- and D . Switching at the target terminal of each K_2^- , we obtain a string. \square

A *necklace* is a signed series-parallel graph obtained as the parallel connection of two strings, at least one of which is nontrivial (see Figure 2).

Lemma 12. *Each reduced signed series-parallel signed graph of depth at least 3 contains a piece switching equivalent to a necklace.*

Proof. Let G be a signed series-parallel graph of depth at least 3. By Observation 5, G contains a piece of depth 3 which is necessarily reduced. We may

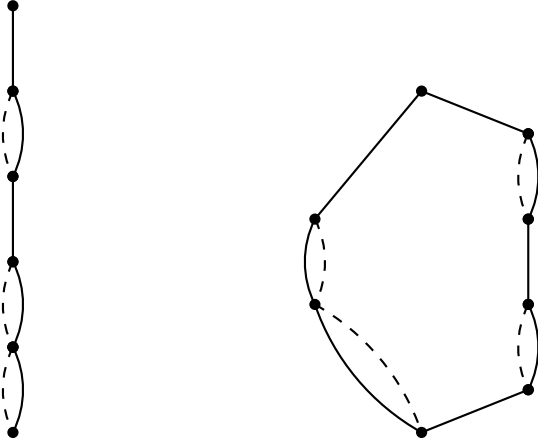


Figure 2: A string (left) and a necklace (right). In this and the following figure, the source terminal is the topmost vertex and the target terminal is the lowermost one.

therefore assume that the depth of G is equal to 3. Since one of its parts has depth 2 and is reduced, it is of series type, so G itself is of parallel type. Lemma 11 implies that G is a parallel connection of graphs switching equivalent to a string. Let H be the parallel connection of two of these graphs, say H_1 and H_2 , where the depth of H_1 equals 2. Note that H_1 has more than two vertices. We show that H is switching equivalent to a necklace. First, if H_2 is a K_2^- , then we perform a switch at one of the terminals to change its sign. Each of the remaining negative edges has an endvertex contained in an unbalanced 2-cycle; we switch at each such endvertex to obtain a necklace. \square

4 Pseudoflows

For the proof of Theorem 3, we utilise the concept of *pseudoflow* in a signed series-parallel graph H , defined just as a nowhere-zero 6-flow in H , except that at each terminal, the inflow is not required to equal the outflow. (In particular, pseudoflows are nowhere-zero by definition.) Let $I_5 = \{-5, -4, \dots, 5\}$. A pseudoflow in H is an (a, b) -pseudoflow (where $a, b \in I_5$) if the outflow at the source terminal equals a and the inflow at the target terminal equals b . An example of a pseudoflow is shown in Figure 3. As another example, note that K_2^+ admits an (a, b) -pseudoflow if and only if $a = b \neq 0$ and $a \in I_5$.

We make a couple of observations related to pseudoflows. An (a, b) -pseudoflow can only exist if a and b have the same parity. A $(0, 0)$ -pseudoflow coincides with a nowhere-zero 6-flow. Furthermore, if the source terminal of H has degree 1, then H admits no $(0, b)$ -pseudoflow for any b . Based on the last observation, let us say that the pair (a, b) is *valid* for $H = (H, s, t)$ if either $a \neq 0$ or $d(s) \geq 2$,

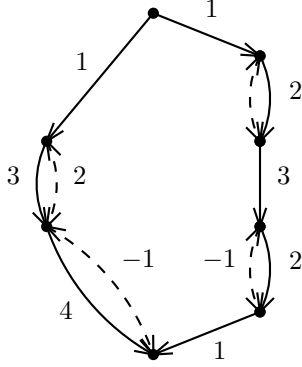


Figure 3: A $(2, 4)$ -pseudoflow in a signed series-parallel graph.

and at the same time $b \neq 0$ or $d(t) \geq 2$.

Observation 13. *Let $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{2}$. If the unbalanced 2-cycle D admits an (a, b) -pseudoflow, then $a \neq \pm b$. In the converse direction, if $a \neq \pm b$ and $a, b \in I_5$, then D admits an (a, b) -pseudoflow.*

The following lemma provides us with information on the types of pseudoflows that exist in strings. For a graph G , we define $\beta(G)$ as the number of distinct 2-cycles in G . Note that $\beta(G) \geq 1$ for any nontrivial string G .

Lemma 14. *Let G be a nontrivial string. Let $a, b \in I_5$ be integers such that $a \equiv b \pmod{2}$ and (a, b) is valid for G . Then G admits an (a, b) -pseudoflow if one of the following conditions holds:*

- (a) $\beta(G)$ is odd and $a \neq \pm b$,
- (b) $\beta(G)$ is even and $a = \pm b$,
- (c) $\beta(G) \geq 2$ and either a is odd or $a = 0$ or $b = 0$.

Proof. Let $n = \beta(G)$ and let the 2-cycles of G be D_1, \dots, D_n . Orient the positive edges of G so as to obtain a directed path from the source to the target terminal. We claim that it suffices to find a sequence of numbers $c_1, \dots, c_{n+1} \in I_5$, all of the same parity, such that $a = c_1$, $b = c_{n+1}$, $c_i \neq \pm c_{i+1}$ for all $i = 1, \dots, n$, and $c_i \neq 0$ unless $i \in \{1, n+1\}$. Given such a sequence, we construct an (a, b) -pseudoflow on G as follows:

- using Observation 13, we find a (c_i, c_{i+1}) -pseudoflow on each D_i , where $1 \leq i \leq n$,
- we assign flow value c_{i+1} to a bridge joining D_i to D_{i+1} if there is one ($1 \leq i \leq n$), with respect to the fixed orientation,

- a bridge incident with a source (target) terminal, if such a bridge exists, will be assigned value $a = c_1$ ($b = c_{n+1}$, respectively) with respect to the fixed orientation.

It is not hard to see that this procedure defines an (a, b) -pseudoflow on G . Note that the values associated to any bridges incident with terminals are nonzero since the pair (a, b) is valid for G .

To find a sequence as above, we consider the possible cases one by one. In cases (a) and (b), we just take the alternating sequence a, b, a, b, \dots of length $n+1$. Consider case (c) and assume first that a is odd. Let d be an odd element of I_5 such that $\pm a \neq d \neq \pm b$. If n is odd, the alternating sequence a, b, \dots of length $n+1$ ends with b as required; otherwise, we insert d after its first element and delete the last element, again obtaining a sequence with the required property.

To finish the discussion of case (c), assume that $a = 0$. Let d be an even element of I_5 such that $0 \neq d \neq \pm b$. Depending on the parity of n , we take either the alternating sequence $0, b, \dots$ of length $n+1$, or the sequence obtained by inserting d after the first 0 and dropping the last element. The case $b = 0$ is symmetric. \square

We use Lemma 14 to obtain a similar result for necklaces:

Lemma 15. *Let G be a necklace and $a, b \in I_5$ integers such that $a \equiv b \pmod{2}$. Then G admits an (a, b) -pseudoflow if either $a \neq \pm b$, or $a = b = 0$ and $\beta(G) \geq 2$.*

Proof. Suppose that G is a parallel connection of strings G_1 and G_2 . Label the strings in such a way that the following conditions hold if possible, in the order of precedence:

- $\beta(G_2) = 0$,
- $\beta(G_2)$ is odd.

We say that G is of *type I* if $\beta(G_2)$ is odd. If $\beta(G_2)$ is even, then G is of *type II*.

Suppose first that $a = b = 0$ and $\beta(G) \geq 2$. If both $\beta(G_1)$ and $\beta(G_2)$ are nonzero, then by Lemma 14 (a) and (c), there is a $(1, 3)$ -pseudoflow in G_1 and a $(-1, -3)$ -pseudoflow in G_2 ; their sum is the required $(0, 0)$ -pseudoflow in G . Suppose then that $\beta(G_2) = 0$, which means that $\beta(G_1) \geq 2$. By Lemma 14 (c), G_1 admits a $(1, 1)$ -pseudoflow f_1 . Since G_2 is necessarily the graph K_2^+ , it admits a $(-1, -1)$ -pseudoflow, which we again sum with f_1 to obtain a $(0, 0)$ -pseudoflow in G .

For the rest of the proof suppose that $a, b \in I_5$ are integers of the same parity such that $a \neq \pm b$. The desired (a, b) -pseudoflow on G will be constructed as a sum of an (a_1, b_1) -pseudoflow on G_1 and an (a_2, b_2) -pseudoflow on G_2 for suitable a_1, b_1, a_2 and b_2 .

Let us consider possible pseudoflows in G_1 . Let $a_1, b_1 \in I_5$ such that $a_1 \equiv b_1 \pmod{2}$. By the choice of G_2 and the fact that at least one string of a necklace is nontrivial, $\beta(G_1) \geq 1$. If $\beta(G_1) = 1$, by Lemma 14 (a), G_1 admits an (a_1, b_1) -pseudoflow if $a_1 \neq \pm b_1$. If $\beta(G_1) \geq 2$, then, by Lemma 14 (c), G_1 admits an (a_1, b_1) -pseudoflow if a_1 and b_1 are odd. In summary, regardless of the type of G_2 , G_1 admits an (a_1, b_1) -pseudoflow if

$$a_1 \text{ and } b_1 \text{ are odd and } a_1 \neq \pm b_1. \quad (1)$$

Next, we consider pseudoflows in G_2 . Let $a_2, b_2 \in I_5$ such that $a_2 \equiv b_2 \pmod{2}$. By Lemma 14 (a) and (b), G_2 admits an (a_2, b_2) -pseudoflow if

$$\text{either } G \text{ is of type I and } a_2 \neq \pm b_2, \text{ or } G \text{ is of type II and } a_2 = \pm b_2. \quad (2)$$

For each possible (a, b) we now exhibit a choice of (a_1, b_1) and (a_2, b_2) satisfying the conditions (1) and (2), respectively, and such that $a_1 + a_2 = a$, $b_1 + b_2 = b$. The (a, b) -pseudoflow in G will be the sum of an (a_1, b_1) -pseudoflow in G_1 and an (a_2, b_2) -pseudoflow in G_2 .

The choices of (a_1, b_1) and (a_2, b_2) are given in Table 1. We assume, without loss of generality, that $|a| \leq |b|$. By inverting all signs if necessary, we may further assume that $a \geq 0$, and if $a = 0$, then $b \geq 0$.

For example, if $(a, b) = (0, 2)$ and G is of type II, then the table suggests taking $(a_1, b_1) = (1, 3)$ and $(a_2, b_2) = (-1, -1)$, in accordance with conditions (1) and (2). The rest of the proof is a routine inspection of the table.

□

Corollary 16. *Every flow-admissible string or necklace admits a $(0, 0)$ -pseudoflow (that is, a nowhere-zero 6-flow).*

5 Proof of Theorem 3

Let G be a counterexample to Theorem 3 with minimum number of edges and, subject to this condition, maximum number of vertices. By Lemmas 8 and 10, G is reduced, and by Corollary 16, its depth is at least 3. Using Lemma 12, we may assume that G contains a piece H that is a necklace. Furthermore, by Corollary 16, $G \neq H$.

We choose H in such a way that $\beta(H)$ is minimized. Recall that $\beta(H)$ is the number of 2-cycles in H , and that $\beta(H) \geq 1$.

Case A: H is not an endpoint of G .

We replace H with $D' = \mathcal{S}(K_2^+, D, K_2^+)$ in G , where D is the unbalanced 2-cycle. Let us call the resulting graph G' . The following lemma provides the last missing piece in our argument.

Lemma 17. *If G contains an unbalanced cycle edge-disjoint from H , then G' is flow-admissible.*

	type I		type II	
a	a_1	a_2	a_1	a_2
b	b_1	b_2	b_1	b_2
0	1	-1	1	-1
2	5	-3	3	-1
0	3	-3	1	-1
4	5	-1	5	-1
2	-1	3	1	1
4	5	-1	3	1
2	-1	3	1	1
-4	-5	1	-5	1
1	3	-2	3	-2
3	-1	4	5	-2
1	3	-2	3	-2
-3	1	-4	-1	-2
1	3	-2	-1	2
5	1	4	3	2
1	3	-2	5	-4
-5	-1	-4	-1	-4
3	5	-2	1	2
5	1	4	3	2
3	5	-2	5	-2
-5	-1	-4	-3	-2

Table 1: The pairs (a_1, b_1) and (a_2, b_2) (the first and second column in each field, respectively) for each possible choice of (a, b) and type of the necklace in the proof of Lemma 15. The case $(a, b) = (0, 0)$ is discussed separately.

Proof. Let u' and v' be the source and target terminal, respectively, of the necklace H (as well as of the graph D'). Let D_0 denote the 2-cycle in D' . Consider an arbitrary edge e of G' . We need to show that e is contained in a signed circuit of G' . Suppose the contrary.

Case 1: $e \notin E(D')$.

We first observe that G contains no $u'v'$ -path that contains e and is vertex-disjoint from H except for its endvertices (let us call such a path an e -detour). Indeed, combining such a path with one of the two $u'v'$ -paths in D' would provide us with a balanced cycle of G' containing e .

In particular, each cycle of G containing e is edge-disjoint from H . Since e is not contained in any signed circuit of G' , any such cycle must be unbalanced. On the other hand, since G is flow-admissible, e is contained in a signed circuit B of G , which must therefore be a barbell. Clearly, B is not edge-disjoint from H , for otherwise B is a signed circuit containing e in G' . Let A_1 and A_2 be the unbalanced cycles in B and let Q be the path connecting them.

If e belongs to an unbalanced cycle of B , say A_1 , then A_1 is edge-disjoint from H in G , and thus also from D' in G' . Since G' is connected, there exists a path connecting A_1 and D_0 and therefore also a barbell of G' containing e , a contradiction. Hence we can assume that $e \in Q$.

Note first that A_1 and A_2 are not both edge-disjoint from H , for otherwise Q contains both u' and v' , and replacing the part of Q inside H with a $u'v'$ -path in D' yields a barbell in G' containing e .

Suppose then that A_1 contains an edge of H . We claim that A_2 is edge-disjoint from H . For the sake of a contradiction, assume that A_2 contains an edge of H . If both A_1 and A_2 were contained in H , then a subpath of Q would be an e -detour. On the other hand, at least one A_i has to be contained in H , for otherwise they both contain u' and v' , violating the definition of a barbell.

We may thus assume that A_1 is contained in H and A_2 is not. Since A_2 contains both terminals of H , we have $Q \subseteq H$, which is a contradiction, since $e \notin E(D')$. Therefore A_2 is edge-disjoint from H as claimed. Let us choose a shortest path P in $Q \cup A_1$ connecting A_2 to a terminal of H ; the union of P , A_2 , D_0 and an edge of D' connecting D_0 to an endvertex of P is then a barbell in G' containing e . This finishes the discussion of Case 1.

Case 2: $e \in E(D')$.

We show that G is of parallel type. Otherwise, it would be of series type, its endparts would be unbalanced (Lemma 8(i)) and by Lemma 9, e would be contained in a barbell, which is a contradiction.

The graph G' is also of parallel type. This is clear if H is a (proper) subgraph of one of the parts of G . Otherwise, the two strings forming H are parts of G , and by the assumption that G contains an unbalanced cycle edge-disjoint from H , there are some more parts. Then G' is the parallel connection of these parts with D' .

By symmetry, we may assume that e is not incident with the terminal v' of

D' . Let A be an unbalanced cycle in G edge-disjoint from D' . Since G' is 2-connected, it contains a path R joining u' to A and avoiding v' . The union of D_0 , R , A and the edge connecting D_0 to u' is a barbell containing e . This concludes the proof. \square

We claim that there is indeed an unbalanced cycle of G that is edge-disjoint from H , as required in Lemma 17. This is clear if G is of series type, because every endpart of G is unbalanced (Lemma 8(i)).

Let G be a parallel connection of its parts H_1, \dots, H_k . The necklace H is either a union of two strings $H_i \cup H_j$ for some i and j , or it is a proper subgraph of some H_i . Suppose first the former — say, $H = H_1 \cup H_2$. Then $k \geq 3$, because $G \neq H$. If there exists $i \in \{3, \dots, k\}$ such that H_i contains an unbalanced cycle, we are done. Since G is reduced, each H_i is reduced as well, and thus every H_i is a signed K_2 . If there exist H_i and H_j ($i, j \in \{3, \dots, k\}$) such that they have opposite signs, then $H_i \cup H_j$ is the sought unbalanced cycle edge-disjoint from H . We conclude that $k = 3$, because G is reduced and it does not contain parallel edges of the same sign. If neither H_1 nor H_2 is a K_2^+ , then $H_1 \cup H_3$ is a necklace of G with $\beta(H_1 \cup H_3) < \beta(H)$, which is a contradiction with the choice of H . On the other hand, if H contains a string that is a K_2^+ , then the string forms an unbalanced 2-cycle with H_3 (as G is reduced), and G is a necklace, which is a contradiction with Corollary 16.

Suppose now that H is a proper subgraph of one of the parts of G , say H_1 . By a similar argument as above we conclude that $k = 2$ and H_2 is a signed K_2 . Moreover H_1 is a series connection of (reduced) graphs H_{11}, \dots, H_{1q} for some q . If any of the graphs H_{1i} that does not contain H contains an unbalanced cycle, we are done. Therefore each H_{1i} that does not contain H is a signed K_2 . Since H is a proper subgraph of H_1 , we conclude that at least one of the terminals of G must be of degree 2. But the terminal is not contained in an unbalanced 2-cycle, which is a contradiction with Lemma 8 (c). This proves the claim that G satisfies the hypothesis of Lemma 17 and the graph G' is flow-admissible.

The graph D' , used to obtain G' , has four edges. All necklaces have at least four edges, with $\mathcal{P}(K_2^+, \mathcal{S}(K_2^+, D))$ and its mirror image being the only ones with exactly four. These necklaces, however, have one vertex fewer than D' , so the choice of G implies that G' admits a nowhere-zero 6-flow φ .

The restriction of φ to D_0 (the 2-cycle in D') is an (a, b) -pseudoflow for some a and b . By Observation 13, $a \equiv b \pmod{2}$ and $a \neq \pm b$. Furthermore, since the values of φ on the edges of $D' - E(D_0)$ are $\pm a$ and $\pm b$, we find that a, b are nonzero elements of I_5 . By Lemma 15, H admits an (a, b) -pseudoflow. Combining it with a restriction of φ , we obtain a nowhere-zero 6-flow in G , contradicting the hypothesis.

Case B: H is an endpart of G .

The argument is similar to the argument of Case A, except that we use the graph $\mathcal{S}(K_2^+, D)$ or $\mathcal{S}(D, K_2^+)$ in place of D' (so as to obtain a flow-admissible

graph) and invoke an analogue of Lemma 17. We arrive at a similar contradiction, which concludes the proof.

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